I.S.I. Bangalore — Ist Semestral exam — 2001-2002
B.Math.Hons.IInd Year
Subject : Algebra III — Instructor : B.Sury

Answer the FIRST question and SIX from the rest.
All questions carry 15 marks.
Any score above 100 will be counted as 100.

Q 1.
Let $a, b, c, d \in \mathbf{Z}$. Consider the homomorphism $\theta : \mathbf{Z}^2 \to \mathbf{Z}^2$ defined by $\theta(x, y) = (ax + by, cx + dy)$. Prove that the image of θ is of finite index in $\mathbf{Z}^2$ if, and only if, $ad \neq bc$. In this case, show further that the smallest natural number n such that $(n, 0)$ and $(0, n)$ belong to $Im(\theta)$ is given by $\frac{|ad-bc|}{(a,b,c,d)}$.

Q 2.
If K is any field and $f \in K[X]$ is irreducible, prove that all the roots of f in any algebraic closure of K have the same multiplicity.

Q 3.
Find all natural numbers n such that the angle of n degrees can be constructed by a ruler and compass.
Hint: Use the theorem we proved on which regular n-gons can be constructed.

Q 4.
Prove that, for any n, all irreducible polynomials of degree n over $\mathbf{F}_p$ are factors of $X^{p^n} - X$ in $\mathbf{F}_p[X]$.

Q 5.
Let L/K be any extension and $S \subset L$ any subset. Define what is meant by S being algebraically independent over K. Prove that if S is algebraically independent over K and $x \in L$, then x is algebraic over $K(S)$ if $S \cup \{x\}$ is algebraically dependent over K.

Q 6.
Let α be any algebraic number which is not in $\mathbf{Q}$. Suppose that K is an algebraic extension of $\mathbf{Q}$ which is maximal with respect to the property that $\alpha \notin K$. Show that any finite extension of K is Galois, whose Galois group is cyclic.

P. T. O.

Q 7.
State the multiplicative version of Hilbert's theorem 90. For $\mathbf{C}/\mathbf{R}$*, prove it directly. What does the theorem give for* $\mathbf{Q}(\sqrt{d})/\mathbf{Q}$ *?*

Q 8.
Suppose L/K *is a finite extension such that the group* $G := \{\sigma \in Aut(L) : \sigma(x) = x \forall x \in K\}$ *satisfies* $L^G = K$*. Prove that* L/K *is a Galois extension.*

Q 9.
Show that the polynomial $X^5 - 6X + 3$ *over* $\mathbf{Q}$ *is not solvable by radicals.*